

ON THE SUM OF THE VORONOI POLYTOPE OF THE LATTICE E_6 WITH A ZONOTOPE

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ABSTRACT. A *parallelotope* is a polytope that tiles the space by translation along a lattice. The Voronoi cell $P_V(L)$ of a lattice L is an example of a parallelotope. A parallelotope can be uniquely decomposed as the Minkowski sum of a zone closed parallelotope P and a zonotope $Z(U)$. In this paper we consider the Minkowski sums $P_V(L) + Z(U)$ and we give several criterion for determining when it is a parallelotope. Using this we classify such zonotopes for some highly symmetric lattices.

In the case of the root lattice E_6 , it is possible to give a more geometric description of the admissible sets of vectors U . We found that the set of admissible vectors, called free vectors, is described by the well-known configuration of 27 lines in a cubic. Based on a detailed study of the geometry of $P_V(E_6)$, we give a simple characterization of the configurations of vectors U such that $P_V(E_6) + Z(U)$ is a parallelotope. The enumeration yield 10 maximal families of vectors, which are presented by their description as regular matroids.

1. INTRODUCTION

Let $L = \mathbb{Z}v_1 + \dots + \mathbb{Z}v_n$ be a *lattice* of rank n in Euclidean space given by n independent vectors $(v_i)_{1 \leq i \leq n}$ in \mathbb{R}^n . The *Voronoi cell* (or *Voronoi polytope*) of L is the convex polytope

$$P_V(L) = \{x \in \mathbb{R}^n : \|x\| \leq \|x - v\| \text{ for all } v \in L\}.$$

A *parallelotope* is a polytope whose translation copies under a lattice L fill the space without gaps and intersections by inner points. For a given lattice L , $\{P_V(L) + v\}_{v \in L}$ define a tiling of \mathbb{R}^n and so $P_V(L)$ is a parallelotope.

A parallelotope P is necessarily centrally symmetric and its facets are necessarily centrally symmetric. A *k-belt* of an n -dimensional polytope P , whose facets are centrally symmetric, is a family of k facets F_1, F_2, \dots, F_k such that $F_i \cap F_{i+1}$ and $F_i \cap F_{i-1}$ are antipodal $(n-2)$ -dimensional faces in F_i for $1 \leq i \leq k$, where the indexing i in F_i taken modulo k . A polytope P is a parallelotope if and only if [Ve54, Mu80, Zo96]:

- P is centrally symmetric
- The facets of P are centrally symmetric
- The facets of P are organized into 4- and 6-belts.

A still open conjecture of Voronoi [Vo08] asserts that any parallelotope is affinely equivalent to a Voronoi polytope. Voronoi conjecture has been solved up to dimension 5 [En00]. For a given set U of vectors, the *zonotope* $Z(U)$ is the Minkowski

The first author is supported by the Croatian Ministry of Science, Education and Sport under contract 098-0982705-2707.

sum

$$Z(U) = \sum_{v \in U} [-v, v].$$

Voronoi's conjecture has been proved for zonotopes [ErRy94, Er99].

For a parallelotope P , a vector v is called *free* if the Minkowski sum $P + [-v, v]$ is again a parallelotope. It is proved in [Gr04] that a vector v is free if and only if the segment $[-v, v]$ is parallel to a facet of each 6-belt of P , but the proof was incomplete and we provide a complete proof in Section 2. A parallelotope is called *nonfree* if it has no free vectors; the first known nonfree parallelotope is $P_V(\mathbf{E}_6^*)$ [En98] and it was later proved that the parallelotopes $P_V(\mathbf{D}_m^+)$ are also nonfree [Gr06a]. A parallelotope P is called *zone-closed* if it cannot be expressed as a Minkowski sum $Q + [-v, v]$ for a vector v . Any parallelotope can be expressed as Minkowski sum of a zone-closed parallelotope and a zonotope. A parallelotope P is called *finitely free* if there exists a finite set $\mathcal{F}(P)$ such that any free vector v is collinear to a vector in $\mathcal{F}(P)$. The parallelotope that we consider in this paper are finitely free Voronoi polytope of highly symmetric lattices. But, for example, the lattice \mathbb{Z}^n has the Voronoi polytope $[-1/2, 1/2]^n$, which is free for any vector v , because it has no 6-belts.

For a given zone-closed parallelotope P , we want to find the vector systems U such that $P + Z(U)$ is still a parallelotope. Of course, every vector in U has to be free. Another condition arising from the theory of matroid [DG99, Gr04] is that U has to be unimodular. If we want to apply Venkov's criterion for parallelotope, then we have to determine the faces of the sum $P + Z(U)$ and the k -belts. If G is a face of P , then we decompose U into $U_1(G) \cup U_2(G) \cup U_3(G)$. The vectors in $U_1(G)$ translate G by some vector $w = \sum_{u \in U_1(G)} \pm u$. The vectors $u \in U_2(G)$ belong to the vector space defined by G and extend G to a larger face. The vectors $u \in U_3(G)$ are *strongly transversal* to P , that is $\dim G + [-u, u] = 1 + \dim G$. Summarizing, we obtain that G correspond to the face

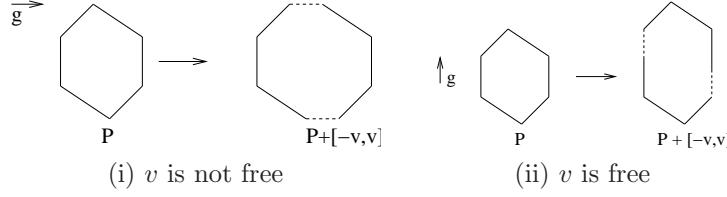
$$G_U = G + w + Z(U_2(G)) + Z(U_3(G)).$$

If $\dim G_U < n - 1$, then G_U cannot be a facet of $P + Z(U)$. If $\dim G_U \leq n - 1$, then G_U is a face of $P + Z(U)$. If $\dim G_U = n$, then since G_U is an n -dimensional polytope we have to compute the facets of G_U in order to get the corresponding facets of $P + Z(U)$.

Hence, facets of $P + Z(U)$ are translated or extended facets of P or obtained from G_U . So, if one computes all facets and $(n - 2)$ -dimensional faces of $P + Z(U)$, then one can determine if it is a parallelotope. The method is explained in Section 2 and applied in Section 6 to the root lattice \mathbf{E}_6 and in Section 8 to other lattices.

The Voronoi polytope $P_V(\mathbf{E}_6)$ of the root lattice \mathbf{E}_6 has only 6-belts. The edges of $P_V(\mathbf{E}_6)$ are of two types: r -edges, which are parallel and equal by norm to minimal vectors of the lattice \mathbf{E}_6 , and m -edges, which are parallel and equal by norm to minimal vectors of the dual lattice \mathbf{E}_6^* . The set \mathcal{R} of minimal vectors of \mathbf{E}_6 is the root system \mathbf{E}_6 . Roots of \mathcal{R} are also facet vectors of $P_V(\mathbf{E}_6)$. The set \mathcal{M} of all, up to sign, 27 minimal vectors of \mathbf{E}_6^* has the combinatorial configuration of the well-known 27 straight lines on a smooth non-degenerate cubic surface in the 3-dimensional projective space. The pairwise scalar products $q^T q'$ of minimal vectors $q, q' \in \mathcal{M}$ take only two values $-\frac{2}{3}$ and $\frac{1}{3}$.

For a lattice L we define an *empty sphere* to be a ball containing no lattice point in its interior. The convex hull of the lattice points on the surface of the ball is

FIGURE 1. The image of a 6-belt of P in $P + [-v, v]$

called a *Delaunay polytope*. The Delaunay polytopes define a tessellation, which is dual to the one by Voronoi polytopes $P_V(L)$, in particular the vertices of $P_V(L)$ are centers of Delaunay polytopes (see, for example, [Sch09]). It turns out that the Delaunay polytopes of E_6 are Schläfli polytope P_{Sch} and its antipodal P_{Sch}^* . The configuration of vectors \mathcal{M} is affinely equivalent to the vertex-set of P_{Sch} .

Instead of 4-belts $P_V(E_6)$ has quasi 4-belts (see definition in Section 4) formed by two pairs of parallel facets such that non-parallel facets intersect by an m -edge. We prove that the set of free vectors of E_6 is exactly \mathcal{M} .

We consider here unimodular subsets $U \subseteq \mathcal{M}$ and give a necessary and sufficient condition for $P_V(E_6) + Z(U)$ to be a parallelotope. Namely, the subset U should be *feasible*, i.e., it should not contain a *forbidden* subset of five vectors $u_i \in \mathcal{M}$, $1 \leq i \leq 5$, such that $u_i^T u_j = \frac{1}{3}$ for $1 \leq i < j \leq 5$. One necessary condition for the sum $P + Z(U)$ of a parallelotope P and a zonotope $Z(U)$ to be a parallelotope is that the set U should span a *unimodular* family of vectors. Thus we prove that subsets $U \subseteq \mathcal{M}$ that do not contain forbidden subsets are unimodular. We found all ten maximal by inclusion feasible subsets $U \subseteq \mathcal{M}$ by a computer search. We give a detailed description of these subsets and regular matroids, represented by these subsets.

We use Coxeter's notations α_n , β_n and γ_n for a regular n -dimensional simplex, cross-polytope and cube, respectively. We also write $h\gamma_n$ for the half cube, i.e. the polytope which is the convex hull of $\{x \in \{0, 1\}^n \text{ s.t. } \sum_{i=1}^n x_i \equiv 0 \pmod{2}\}$. In Section 3, a construction of the root lattice E_6 is given, followed in Section 4 by the Schläfli polytope which is the Delaunay polytope of E_6 . Then in Section 5 the Voronoi polytope of E_6 is given with the description of the free vectors. In Section 6, a criterion for a vector system to be feasible is given and in Section 7 the enumeration of the 10 maximal feasible systems is given. In Section 8, we consider how such results can be extended to other lattices.

2. GENERAL RESULTS AND ENUMERATION ALGORITHMS

Below we give a characterization of free vectors that can actually be used to enumerate them on computer or in case by case analysis as below for E_6 .

Theorem 1. ([Gr04]) *Let P be a parallelotope and v be a vector. The following assertions are equivalent:*

- (i) P is free along v (i.e. $P + [-v, v]$ is a parallelotope);
- (ii) $v^T p = 0$ for at least one facet vector p of each 6-belt of P .

Proof. If v does not satisfy (ii) then Figure 1.(i) shows that there is a 8-belt in $P + [-v, v]$. So, (ii) is necessary.

Let us now assume that (ii) is true. Since P and $[-v, v]$ are centrally symmetric, the Minkowski sum $P + [-v, v]$ is as well. If F is a facet of $P + [-v, v]$ which is a translate of a facet of P , then it is necessarily centrally symmetric. If F is a facet of $P + [-v, v]$ which is not such a translate, then it is of the form $G + [-v, v]$ with G an $(n-1)$ or $(n-2)$ -dimensional face of P . If G is $(n-1)$ -dimensional, then it is necessarily centrally symmetric. If G is $(n-2)$ -dimensional, then it is necessarily part of a 4-belt, and such $(n-2)$ -dimensional faces are centrally symmetric (see [Gr04] for details).

Let $\{F_i : 1 \leq i \leq 2k\}$ be a $2k$ -belt of $P + [-v, v]$. Here F_i are facets, whose facet vectors are orthogonal to the vector v . For $1 \leq i \leq 2k$, $F_i = F'_i + [-v, v]$, where F'_i is a facet of P or $F_i = G_i + [-v, v]$, where G_i is an $(n-2)$ -face of P . The face G_i is an intersection of two facets of a 4-belt of P . Let $F'_i = F'_i$ if $F_i = F'_i + [-v, v]$ and $F'_i = G_i$ if $F_i = G_i + [-v, v]$.

Let c_i , $1 \leq i \leq 2k$, be a vector with an end-vertex in the center of F_i . The $2k$ end-points of c_i lie on a two-dimensional plane K , since these $2k$ points are centers of a $2k$ -belt. Since F'_i is centrally symmetric, the centers of F_i and F'_i coincide. Hence the vectors $2c_i$ are lattice vectors of the lattice of the parallelotope P . The intersection $P \cap K$ is a $2k$ -gon.

Let P_i be the translation copy of P by the vector $2c_i$. The parallelotope P_i is contiguous to P by the face F'_i . The intersection of K with P and all P_i gives a $2k$ -gon surrounded by $2k$ non-intersecting $2k$ -gons. It is possible only if $k \leq 3$. \square

Since we do not know whether Voronoi's conjecture is true or not, it can happen in principle that P is a Voronoi polytope for a lattice L and $P + [-u, u]$ is a parallelotope but not a Voronoi polytope. Any facet of P is defined by a vector $u \neq 0$ of the lattice L and in [Gr06b] it is proved that $P + [-v, v]$ is a Voronoi polytope if and only if v is free and all vectors u defining facets of P have scalar product 0, $-a$ and a with v for some $a \in \mathbb{R}$.

Let us define \mathcal{S}^n , $\mathcal{S}_{\geq 0}^n$ and $\mathcal{S}_{> 0}^n$ the spaces of symmetric, semidefinite symmetric and positive definite symmetric matrices, respectively. If one considers more than one lattice, then the viewpoint of quadratic forms is useful. That is we replace L by \mathbb{Z}^n and the quadratic form $\|x\|^2$ by a quadratic form $x^T A x$ with $A \in \mathcal{S}_{> 0}^n$. This builds a dictionary between lattices and positive definite quadratic forms. Given a polytope P with vertex-set in \mathbb{Z}^n the condition that it is a Delaunay polytope for the norm $x^T A x$ is translated into linear equalities and strict inequalities on the coefficients of A . Let us fix a Delaunay tessellation of \mathbb{Z}^n by $A_0 \in \mathcal{S}_{> 0}^n$. The condition that $A \in \mathcal{S}_{> 0}^n$ determines the same Delaunay tessellation as A_0 determines a convex cone in $\mathcal{S}_{> 0}^n$, called L -domain. Voronoi proved that L -domains are polyhedral and that $\mathcal{S}_{> 0}^n$ is partitioned by them (see [Vo08, Sch09] for more details).

A lattice is called *rigid* if the corresponding L -domain is of dimension 1. Actually [Sch09] for a given extreme ray A_i of an L -domain we can define a corresponding Delaunay tessellation and Voronoi polytope. In [DV05] it is proved that if V_A is a Voronoi polytope of a positive definite matrix A , then we have

$$V_A = \sum_i V_{A_i}$$

with A_i the extreme rays of the L -domain in which A is contained. For example, a Voronoi polytope is a zonotope if and only if the extreme rays of its L -domain are

of rank 1 [ErRy94]. Suppose that we have a basis $\mathbf{e} = \{e_1, \dots, e_n\}$ of L , and L has Gram matrix A relative to \mathbf{e} . For a vector $u \in \mathbb{R}^n$ the rank 1 form $p(u) = qq^T$ is the form defined by the expression of u in the dual basis \mathbf{e}^* if $u = q_1 e_1^* + \dots + q_n e_n^*$. If $U = \{u_1, \dots, u_m\}$ then $P + Z(U)$ is a Voronoi polytope if and only if the cone

$$LT(A, U) = \mathbb{R}_+ A + \mathbb{R}_+ p(u_1) + \dots + \mathbb{R}_+ p(u_m)$$

is an L -domain.

For highly symmetric lattices, some efficient techniques for computing their Delaunay tessellation and so by duality Voronoi polytopes have been introduced in [DSV09]. These techniques use the quadratic form viewpoint for the actual computation. Let us now consider the used algorithms for enumerating free vectors and strongly transversal faces.

For the enumeration of free vectors, Theorem 1 gives implicitly a method for enumerating them. The first step is to use [DSV09] in order to get the Delaunay tessellation and, as a consequence $P_V(L)$. From this we get the list of 6-belts of L by Theorem 1. Free vectors must satisfy an orthogonality condition for each 6-belt. So, if we have N 6-belts, then we have 3^N cases to consider, which can be large. The enumeration technique consider the 6-belts one by one by making choice at each step. We use symmetries to only keep non-isomorphic representatives of all choices. We also use the fact that any choice among the three possibilities implies a linear equality on the coefficients of the free vector v . Hence the choice made for some 6-belts might imply other choices for other 6-belts. So, the dimension decreases at each step and the number of choices is thus only 3^n at most. At the end we have a number of vector spaces containing the free vectors. $P_V(L)$ is nonfree, respectively finitely free, if and only if all the vector spaces are 0-dimensional, respectively at most 1-dimensional.

The enumeration of strongly transversal faces can be done in the following way: If G is a strongly transversal face of P , then any subface of P is also strongly transversal. Thus starting from the vertices of P , which correspond to Delaunay polytopes of P , we can enumerate all strongly transversal faces of P and hence describe the facet and belt structure of $P + Z(U)$ with U a set of free vectors. By using Venkov's condition this allows to determine whether or not $P + Z(U)$ is a parallelotope.

Another variant is to write $U = \{u_1, \dots, u_p\}$ and write

$$P + Z(U) = P' + [-u_p, u_p] \text{ with } P' = P + Z(\{u_1, \dots, u_{p-1}\}).$$

If we know that P' is a Voronoi polytope, then we can test whether $P + Z(U)$ is a parallelotope by testing whether u_p is a free vector via Theorem 1. Furthermore, by using the sign condition from [Gr06b] mentioned above we can test whether $P + Z(U)$ is a Voronoi polytope. The method relies on computing the Delaunay tessellation.

Yet another method is to compute the polyhedral cone $L(A, U)$ and check if it defines an L -domain. The method is to take a point in the relative interior of $L(A, U)$, computing the Delaunay tessellation by using the methods of [DSV09] and then the facets of the corresponding L -domain by using the method explained in [DSV08] and to check that it coincides with $L(A, U)$. It is best to choose the point inside $L(A, U)$ to be the sum $A + p(u_1) + \dots + p(u_m)$ so that we can use the symmetry of the configuration U in the computation of the Delaunay tessellation.

We choose to use the second method because it allows us to distinguish between parallelotope and Voronoi polytope. The process of enumeration is then done by considering all subsets U of the set of free vectors and adding vectors one by one and testing whether they are feasible or not. By doing so we are able to get the list $\mathcal{F}_{min}(L)$ of minimal forbidden subsets of $\mathcal{F}(L)$ and the list $\mathcal{F}_{max}(L)$ of maximal feasible subsets. The cost of computing the Delaunay tessellation is relatively expensive, hence we always use the list of already known forbidden subsets in order to avoid such computation whenever possible. We found out that in the case that we consider in Section 8, whenever a polytope $P_V(L) + Z(U)$ is a parallelotope then it is also a Voronoi polytope, thereby confirming Voronoi's conjecture in those cases.

However, for E_6 we prefer to use the first enumeration method.

3. 27 LINES

There are exactly 27 straight lines on any smooth non-degenerate cubic surface in the 3-dimensional projective space. This fact is very well known (see, for example, [Cox83], and other papers of Coxeter). It is proved in many textbooks on algebraic geometry (see, for example, [Re88, ch.3, §7], [Sha88, ch.IV, §2.5]). The combinatorial configuration of the set \mathcal{L} of these 27 lines is unique. For example, if two lines $l, l' \in \mathcal{L}$ intersect, then there is a unique line $l'' \in \mathcal{L}$ that intersects both lines l, l' . Every three mutually intersecting lines generate a tangent plane. Each line intersects exactly 10 lines and belongs to exactly 5 of all 45 tangent planes.

Schläfli described his famous *double six*

$$\begin{array}{cccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{array}$$

which is a special arrangement of twelve lines from \mathcal{L} on the surface such that any two of them intersect if and only if they occur in different rows and different columns. Any two columns determine a pair of planes $a_i b_j$ and $a_j b_i$ whose intersection $c_{ij} = c_{ji}$ intersects all the four lines and therefore must lie entirely on the surface. In this way Schläfli obtained his notations a_i, b_j, c_{ij} for all 27 lines.

Burnside [Burn11, pp.485–488], used these symbols of 27 lines as elements of an algebra (in fact as vectors in 6-dimensional space). This amounts to representing the lines by 27 points in an affine 6-space, such that the 45 triangles representing the tangent planes all have the same centroid. Using this centroid as the origin, he applies Schläfli's symbols for the 27 lines to the positions of vectors of the 27 points, so that

$$(1) \quad a_i + b_j + c_{ij} = 0, \quad c_{ij} + c_{kl} + c_{mn} = 0, \quad \text{where } \{ijklmn\} = \{123456\} = N_6.$$

Choose as a basis \mathcal{A} the set of the six vectors a_1, \dots, a_6 , representing six pairwise skew lines. Let us define

$$h = \frac{1}{3} \sum_{i \in N_6} a_i.$$

Then the remaining 21 vectors are $b_i = a_i - h$, $c_{ij} = h - a_i - a_j$.

The set \mathcal{M} of 27 vectors a_i, b_j, c_{kl} represents the combinatorial configuration of the set \mathcal{L} of the 27 lines by a set \mathcal{M}' of 27 lines in the real 6-dimensional space \mathbb{R}^6 , all lines intersect in the origin. Let ϕ_i , $i = 1, 2$, be the two possible values of angles $\phi(m, m')$ between two lines $m, m' \in \mathcal{M}'$. This representation is such that $\phi(m, m') = \phi_1 = \text{Arcos}\frac{1}{2}$ if the lines $l, l' \in \mathcal{L}$ intersect, and $\phi(m, m') = \phi_2 = \text{Arcos}\frac{1}{4}$, otherwise.

$\dim G$	0	1	2	3	4	5	6
type of G	vertex α_0	edge α_1	α_2	α_3	α_4	α_5 or β_5	P_{Schl}
$n(G)$	27	216	720	1080	$432 + 216$	$72 + 27$	1

TABLE 1. Faces G of the Schläfli polytope P_{Schl} .

It is convenient to choose norms a_i^2 of basic vectors equal to $\frac{4}{3}$. In this case the scalar product of basic vectors a_i and a_j is the same for all $i \neq j$:

$$a_i^T a_j = a_i^2 \cos \phi_2 = \frac{4}{3} \cdot \frac{1}{4} = \frac{1}{3}.$$

Hence, the elements a_{ij} of the Gram matrix A of the basis \mathcal{A} take only two values: $a_{ii} = \frac{4}{3}$, $a_{ij} = a_{ji} = \frac{1}{3}$ if $i \neq j$. Besides, norms of all vectors $q \in \mathcal{M}$ are the same: $q^2 = \frac{4}{3}$.

The lattice integrally generated by the basis \mathcal{A} and the vector h is E_6^* which is dual to the root lattice E_6 . This representation of the lattice E_6^* was used by Baranovskii in [Ba91] for a description of Delaunay polytopes of E_6^* . Barnes uses in Formula (8.5) of [Barn57] the basis \mathcal{A} for to describe minimal vectors of E_6^* . Vectors of E_6^* have all coordinates in the basis \mathcal{A} equal to one third of an integer. The 27 vectors of the set

$$(2) \quad \mathcal{M} = \{a_i, b_i = a_i - h, c_{ij} = h - a_i - a_j, \text{ where } 1 \leq i < j \leq 6\},$$

are, up to sign, the 27 minimal vectors of the lattice E_6^* . The lattice E_6 has an automorphism group equal to $W(E_6) \times \{\pm Id_6\}$ with $W(E_6)$ being the Weyl group of E_6 (see [Hu90] for more details), which is also the automorphism group of \mathcal{M} .

The inverse matrix $E = A^{-1}$, which is the Gram matrix of the dual basis $\mathcal{E} = \{e_i : i \in N_6\}$, has elements

$$e_{ii} = e_i^2 = \frac{8}{9}, \quad i \in N_6, \quad e_{ij} = e_{ji} = e_i^T e_j = -\frac{1}{9}, \quad i \neq j.$$

It is easy to check that vectors of the dual lattice E_6 have in the basis \mathcal{E} the following form

$$\sum_{i \in N_6} z_i e_i, \text{ where } z_i \in \mathbb{Z} \text{ and } \sum_{i \in N_6} z_i \equiv 0 \pmod{3}.$$

One can show that minimal vectors of E_6 in the basis \mathcal{E} form a set \mathcal{R} of 72 vectors, which, up to sign, have the form:

- 15 vectors of the form $e_i - e_j$, $1 \leq i < j \leq 6$;
- 20 vectors of the form $e(S) = e_i + e_j + e_k$, where $S = \{ijk\} \subset N_6$;
- 1 vector of the form $e(N_6) = \sum_{i \in N_6} e_i$.

It is easy to verify that, for $r, r' \in \mathcal{R}$, $r' \neq r$, we have $r^2 = 2$ and $r^T r' \in \{0, \pm 1\}$. This means that these vectors form the root system E_6 .

4. THE SCHLÄFLI POLYTOPE P_{Schl}

The convex hull of end-points of all vectors of the set \mathcal{M} is the Schläfli polytope P_{Schl} , i.e., $P_{Schl} = \text{conv } \mathcal{M}$. Since P_{Schl} is a Delaunay polytope of the lattice E_6 and the Voronoi polytope $P_V(E_6)$ of the lattice E_6 is the convex hull of P_{Schl} and its centrally symmetric copy P_{Schl}^* , we have to study properties of P_{Schl} .

For $q \in \mathcal{M}$, let $v(q)$ be the corresponding vertex of P_{Schl} . Two vertices $v(q)$ and $v(q')$ are adjacent by an edge $e = (v(q), v(q'))$ if and only if $q^T q' = \frac{1}{3}$, i.e., if the corresponding lines of \mathcal{L} are skew. The norm of the edge e is $e^2 = (q - q')^2 = 2$. The edge e is parallel to a root $r \in \mathcal{R}$, $r = q - q'$. Since P_{Schl} has 216 edges (see, for example, [CS91]) and \mathcal{R} contains, up to sign, 36 roots, all edges are partitioned into 36 classes, each containing six pairwise parallel edges.

The polytope P_{Schl} has two types of facets: 20 pairs of simplicial facets $\alpha_5(r)$ and $\alpha_5(-r)$ which are orthogonal to roots r of the root system \mathcal{R} ; and 27 cross-polytopal facets $\beta_5(q)$ which are orthogonal to the vectors $q \in \mathcal{M}$ and are opposite to vertices $v(q)$. Both the types of facets are regular polytopes with facets α_4 . This implies that all faces of P_{Schl} of dimension $k \leq 4$ are regular simplices α_k .

Sets of vertices of $\beta_5(q)$ and $h\gamma_5(q)$ are the sets $\{v(p) : p \in T(q)\}$ and $\{v(p) : p \in S(q)\}$, where

$$T(q) = \left\{ p \in \mathcal{M} : q^T p = -\frac{2}{3} \right\} \text{ and } S(q) = \left\{ p \in \mathcal{M} : q^T p = \frac{1}{3} \right\}.$$

Let $I_5 = \{1, 2, \dots, 5\}$, and let $v(q_i)$, $v(q'_i)$, $i \in I_5$, be the five pairs of opposite vertices of the facet $\beta_5(q)$. Then

$$T(q) = \{q_i, q'_i : i \in I_5\}.$$

Let $p \in S(q)$. Since $p^T q = \frac{1}{3}$ and $q + q_i + q'_i = 0$, we have $p^T (q_i + q'_i) = -\frac{1}{3}$. This equality and the inclusions $p^T q_i, p^T q'_i \in \{-\frac{2}{3}, \frac{1}{3}\}$ imply that $p^T q_i \neq p^T q'_i$. Hence, the $16 = 2^5$ vectors $p \in S(q)$ determine 16 partitions $T(q) = T_J(q) \cup T_{J'}(q)$, where $J' = I_5 - J$,

$$(3) \quad T_J(q) = \{q_i : i \in J\} \cup \{q'_i : i \in J'\}, \text{ and } J = J(p) = \left\{ i \text{ s.t. } p^T q_i = \frac{1}{3} \right\}.$$

The set $T_J(q)$ contains five linearly independent vectors such that $p^T p' = \frac{1}{3}$ for any $p, p' \in T_J(q)$. Moreover, since $\{v(p) : p \in T(q)\}$ is the set of ten vertices of the cross-polytope $\beta_5(q)$ and any 4-face of $\beta_5(q)$ is a simplex α_4 , the set of vertices of each 4-face of $\beta_5(q)$ has the form $\{v(p) : p \in T_J(q)\}$ for all 32 subsets $J \subseteq I_5$.

Lemma 1. *Any subset $X \subseteq \mathcal{M}$ of cardinality $|X| = 5$ such that $p^T p' = \frac{1}{3}$ for all $p, p' \in X$, $p \neq p'$, is a set $T_J(q)$ for some $q \in \mathcal{M}$ and $J \subseteq I_5$.*

Proof. Recall that $v(p)$ is a vertex of P_{Schl} for any $p \in X$, and $\alpha_4(X) = \{v(p) : p \in X\}$ is the set of vertices of a regular 4-simplex α_4 with all r -edges. Of course, $\alpha_4(X)$ is a 4-face of P_{Schl} . But any 4-face of P_{Schl} is α_4 (see [CS91]). Therefore, $\alpha_4(X)$ is a 4-face either of a facet $\beta_5(q)$ for some $q \in \mathcal{M}$, or of a facet $\alpha_5(r)$ for some $r \in \mathcal{R}$. But each facet of type $\alpha_5(r)$ is contiguous in P_{Schl} by a 4-face only to a facet of type $\beta_5(q)$. This means that each 4-face of P_{Schl} is a 4-face of some facet of type $\beta_5(q)$. The set of vertices of each 4-face of this facet has the form $\{v(p) : p \in T_J(q)\}$ for some $J \subseteq I_5$. \square

Recall that $q_i + q'_i + q = 0$ for all $i \in I_5$. Let $t_i = q_i - q'_i$ be the diagonal of $\beta_5(q)$ connecting the vertices $v(q_i)$ and $v(q'_i)$. The facet $\beta_5(q)$ has five pairwise orthogonal diagonals t_i for $i \in I_5$. Any two non-opposite in $\beta_5(q)$ vertices are adjacent by an r -edge, which is parallel to a root of \mathcal{R} . Denote the root which is parallel to the edge $(v(q_i), v(q_j))$ by $r_{ij} = q_i - q_j$. Since $q_i + q'_i + q = q_j + q'_j + q = 0$, we have also

$r_{ij} = q'_j - q'_i$. Set $r'_{ij} = q_i - q'_j = q_j - q'_i$. It is easy to verify that

$$(4) \quad r_{ij}^T r'_{ij} = 0, \quad r_{ij} + r'_{ij} = t_i, \quad r'_{ij} - r_{ij} = t_j, \quad r_{ij} = \frac{1}{2}(t_i - t_j), \quad r'_{ij} = \frac{1}{2}(t_i + t_j).$$

Since there are ten pairs $i, j \in I_5$ for $i \neq j$, there are, up to sign, 20 roots r_{ij}, r'_{ij} for $i, j \in I_5$ and $i \neq j$. These are all roots that are orthogonal to the vector q . In fact, each vector $q \in \mathcal{M}$ determines a partition of the lattice E_6 into 5-dimensional layers $L_k(q) = \{u \in E_6 : q^T u = k\}$ for all $k \in \mathbb{Z}$. Each layer is isomorphic to the root lattice D_5 . Note that the Delaunay polytopes of D_5 are β_5 (1 translation class) and $h\gamma_5$ (2 translation classes). If a Schläfli polytope intersects a layer, then this intersection is either a point, or one of Delaunay polytopes of the lattice D_5 , namely either $\beta_5(q)$ or a half-cube $h\gamma_5(q)$. The set

$$\mathcal{R}(q) = \{r \in \mathcal{R} : q^T r = 0\}$$

is the set of minimal vectors of each layer which is the root system D_5 containing, up to sign, 20 roots, i.e. $\mathcal{R}(q)$ contains 40 roots.

Besides, for $p \in \mathcal{M} - \{q_i, q'_i\}$, we have

$$(5) \quad t_i^T p \in \{0, \pm 1\} \text{ and } t_i^T q_i = 2, \quad t_i^T q'_i = -2, \quad t_i^2 = 4 \text{ for all } i \in I_5.$$

In fact, recall that $t_i = q_i - q'_i$. For $p \in \mathcal{M}$, if $p^T q_i \neq p^T q'_i$, then $p^T q_i = \frac{1}{3}$ implies $p^T q'_i = -\frac{2}{3}$, and $p^T q_i = -\frac{2}{3}$ implies $p^T q'_i = \frac{1}{3}$.

For any vector $a \in \mathbb{R}^6$, define two hyperplanes

$$H(a) = \{x \in \mathbb{R}^6 : a^T x = 0\} \text{ and } H_s(a) = \left\{x \in \mathbb{R}^6 : a^T x = \frac{1}{2}a^2\right\}.$$

The affine hyperplane $H_s(a)$ is a parallel shift of $H(a)$. For example, for a root $r \in \mathcal{R}$, the hyperplane $H_s(r)$ is supporting for the facet $\alpha_5(r)$. Similarly, for $q \in \mathcal{M}$, the hyperplane $H_s(-q)$ supports the facet $\beta_5(q)$.

Let $T_i(q) = T(q) \cup \{q\} - \{q_i, q'_i\}$ and $T_{ij}(q) = T_i(q) \cap T_j(q)$.

Lemma 2. *Let t_i be a diagonal of the facet $\beta_5(q)$ for some $q \in \mathcal{M}$. Then $\mathcal{M} \cap H(t_i) = T_i(q)$.*

Proof. It is easy to verify that $T_i(q) \subseteq \mathcal{M} \cap H(t_i)$. Recall that $t_i = q_i - q'_i$. Consider the three facets $\beta_5(q)$, $\beta_5(q_i)$ and $\beta_5(q'_i)$. The three vertices $v(q)$, $v(q_i)$ and $v(q'_i)$ form a regular triangle Δ with edges $s'_i = (v(q), v(q_i))$, $t_i = (v(q_i), v(q'_i))$ and $s_i = (v(q'_i), v(q))$. The three edges s'_i, t_i, s_i are diagonals of the facets $\beta_5(q'_i), \beta_5(q)$ and $\beta_5(q_i)$, respectively. Each of the hyperplanes $H(s'_i), H(t_i)$ and $H(s_i)$ contains a subset of nine vectors of \mathcal{M} of type $T_k(p)$ for $p = q'_i, q, q_i$, respectively. The intersection $G = H(t_i) \cap H(s_i) \cap H(s'_i)$ is 4-dimensional. It is orthogonal to the 2-plane of the triangle Δ , where the vectors q, q_i, q'_i lie. Hence, $G \cap \mathcal{M} = \emptyset$, since \mathcal{M} does not contain pairs of orthogonal vectors, and G is orthogonal to the vectors q, q_i, q'_i . This implies that the set \mathcal{M} is partitioned in three non-intersecting sets $\mathcal{M} \cap H(t_i)$, $\mathcal{M} \cap H(s_i)$, $\mathcal{M} \cap H(s'_i)$. These sets are of type $T_k(p)$ for $p = q, q_i, q'_i$, and each $T_k(p)$ contains 9 vectors. \square

Lemma 2 and formula (4) allow to prove the following

Lemma 3.

$$H(t_i) \cap H(t_j) \cap \mathcal{M} = H(r_{ij}) \cap H(r'_{ij}) \cap \mathcal{M} = T_{ij}(q).$$

Proof. The relations in (4) $r_{ij} = \frac{1}{2}(t_i - t_j)$ and $r'_{ij} = \frac{1}{2}(t_i + t_j)$ imply $H(r_{ij}) \cap H(r'_{ij}) = H(t_i) \cap H(t_j)$. Now Lemma 2 implies the assertion of this Lemma. \square

$\dim G$	type of G	$(k-1)$ -subfaces	$n(G)$
0	vertex	–	54
1	r -edge	2 vertices	432
1	m -edge	2 vertices	270
2	triangle	1 r -edge 2 m -edges	2160
3	tetrahedron	4 triangles	2160
4	V_{22}^4	9 tetrahedra	720
5	$\text{diplo}(\alpha_5)$	20 V_{22}^4	72
6	$P_V(\mathbb{E}_6)$	72 $\text{diplo}(\alpha_5)$	1

TABLE 2. Faces G of the Voronoi polytope $P_V(\mathbb{E}_6)$.5. THE VORONOI POLYTOPE $P_V(\mathbb{E}_6)$

Let the origin be the center of P_{Schl} . Then the convex hull of P_{Schl} and its centrally symmetric copy P_{Schl}^* is the Voronoi polytope of the lattice \mathbb{E}_6 , i.e.,

$$P_V(\mathbb{E}_6) = \text{conv}(P_{Schl} \cup P_{Schl}^*) = \text{diplo}(P_{Schl}),$$

(see [CS91], where the notation $\text{conv}(P \cup P^*) = \text{diplo}(P)$ is introduced). One can prove that the intersection $P_{Schl} \cap P_{Schl}^*$ is the Voronoi polytope of the dual lattice \mathbb{E}_6^* .

The convex hull $\text{conv}(P_{Schl} \cup P_{Schl}^*) = P_V(\mathbb{E}_6)$ is such that $P_V(\mathbb{E}_6)$ has 27 pairs of opposite vertices $v(q), -v(q) = v(-q)$ for $q \in \mathcal{M}$, and 36 pairs of parallel opposite facets $F(r) = \text{conv}(\alpha_5(r) \cup \alpha_5^*(r)) = \text{diplo}(\alpha_5)$ and $F(-r) = -F(r)$. Here α_5^* is the centrally symmetric copy of α_5 . The simplex α_5^* is a facet of P_{Schl}^* . The 12 vertices $v(p_i)$ of $F(r)$ are adjacent by 12 r -edges to their 12 shifted copies $v(p_i - r)$ in $-F(r)$.

Recall that edges of $P_V(\mathbb{E}_6)$ are of two types: r - and m -edges. Vertices $v(q)$ and $v(q')$ are adjacent by an r -edge if and only if $q^T q' = \frac{1}{3}$, and then either $q, q' \in \mathcal{M}$ or $q, q' \in -\mathcal{M}$. The r -edges belong either to P_{Schl} or to P_{Schl}^* . Each m -edge connects a vertex $v(q)$ of P_{Schl} to a vertex $v(-q')$ of P_{Schl}^* for $q, q' \in \mathcal{M}$. These vertices are adjacent by an m -edge if and only if $q^T q' = -\frac{2}{3}$. Hence, the norm of the m -edge $(v(q), v(-q'))$ is equal to the norm $(q + q')^2 = \frac{4}{3}$ of the vector $-(q + q') \in \mathcal{M}$.

The r -edges of a facet $F = \text{diplo}(\alpha_5)$ of $P_V(\mathbb{E}_6)$ are edges of the regular simplices α_5 and α_5^* . All the 30 r -edges are partitioned into 15 pairs of parallel edges. A vertex v of α_5 is adjacent by an m -edge to a vertex v' of α_5^* if and only if v' is not the vertex v^* , which is opposite to v in F . All the 30 m -edges of F are partitioned into 15 pairs of parallel m -edges.

Two facets $F(r)$ and $F(r')$ of $P_V(\mathbb{E}_6)$ intersect by a 4-face if and only if $r^T r' = 1$. In this case $r'' = r - r'$ is a root, and $r^T r'' = -(r')^T r'' = 1$. Hence, the six facets $F(\pm r), F(\pm r'), F(\pm r'')$ form a 6-belt. The Voronoi polytope $P_V(\mathbb{E}_6)$ has 120 6-belts.

If $r^T r' = 0$, then the intersection $F(r) \cap F(r')$ consists only of one m -edge, and we say that the four facets $F(\pm r), F(\pm r')$ form a *quasi 4-belt*. Each quasi 4-belt determines uniquely a set of four pairwise parallel m -edges which are the intersections of the facets of the quasi 4-belt.

Each facet $F(r)$ is the convex hull of any two opposite 4-faces. Each 4-face of $P_V(E_6)$ has the type of a four-dimensional repartitioning polytope V_{22}^4 . This polytope is the convex hull of two regular triangles with r -edges. The 2-planes spanned by these triangles are mutually orthogonal and intersect in the common center of both triangles. Every pair of vertices of distinct triangles is adjacent by an m -edge.

Each 3-face is a tetrahedron with two non-adjacent r -edges and four m -edges. Each 2-face is a triangle with two m -edges and one r -edge.

All m -edges are partitioned into 27 classes $E(q)$ of edges parallel to vectors $q \in \mathcal{M}$. Each class $E(q)$ contains 10 edges. Each of ten edges of $E(q)$ connects a vertex $v(q_i)$ of the facet $\beta_5(q)$ of P_{Schl} with the vertex $v(-q'_i)$ of the facet $\beta_5(-q)$ of P_{Schl}^* . Hence, the m -edges of $E(q)$ are partitioned into five pairs $e_i, -e_i$ of opposite in $P_V(E_6)$ edges $e_i = (v(q_i), v(-q'_i))$ and $-e_i = (v(-q_i), v(q'_i))$, i.e.,

$$E(q) = \{\pm e_i : i \in I_5\}.$$

Call a pair of edges $e, e' \in E(q)$ *proper* if $e' \neq \pm e$. Let $E^2(q)$ be the set of all proper pairs of $E(q)$. The set $E^2(q)$ contains $40 = 2^2 \binom{5}{2}$ proper pairs.

Proposition 1. *Each proper pair of $E^2(q)$ is contained in a unique facet $F(r)$ for $r \in \mathcal{R}(q)$. There is a one-to-one correspondence $f : E^2(q) \rightarrow \mathcal{R}(q)$ with the following property. If $r_{ij} = f(e_i, e_j)$ and $r'_{ij} = f(e_i, -e_j)$, $i, j \in I_5$, then the roots r_{ij} and r'_{ij} satisfy the equalities (4).*

Proof. Recall that each facet of $P_V(E_6)$ contains 15 pairs of parallel m -edges. Each such pair is proper, since a facet does not contain two opposite in $P_V(E_6)$ edges. Recall also that each 4-face has the type of the repartitioning polytope V_{22}^4 , which does not contains pairs of parallel edges. This implies that a proper pair can be contained in at most one facet. All the 72 facets of $P_V(E_6)$ contain $72 \times 15 = 27 \times 40$ distinct proper pairs. But this is just the number of all proper pairs. This implies that every proper pair (e, e') is contained in a uniquely determined facet $F(r)$, i.e. $f(e, e') = r$.

Obviously, if $(e, e') \in E^2(q)$, then $r = f(e, e') \in \mathcal{R}(q)$. Recall that $\mathcal{R}(q)$ contains 40 roots. This implies that f is a one-to-one correspondence between $E^2(q)$ and $\mathcal{R}(q)$.

Consider the facets $F(r_{ij})$ and $F(r'_{ij})$, where $r_{ij} = f(e_i, e_j)$, $r'_{ij} = f(e_i, -e_j)$ and $(e_i, e_j), (e_i, -e_j) \in E^2(q)$, $e_i = (v(q_i), v(-q'_i))$, $e_j = (v(q_j), v(-q'_j))$. The facet $F(-r_{ij})$, which is opposite in $P_V(E_6)$ to the facet $F(r_{ij})$, contains the opposite edges $-e_i, -e_j$. Since this pair of edges is proper, $f(-e_i, -e_j) = -r_{ij}$. Among the four facets $F(\pm r_{ij}), F(\pm r'_{ij})$ any two non-opposite facets have a non-empty intersection, containing an m -edge of $E(q)$. It is possible only if the four facets form a quasi 4-belt. Hence, $r_{ij}^T r'_{ij} = 0$. The end-vertices $v(q_i)$ and $v(-q'_i)$ of the edge e_i belong to the intersection $F(r_{ij}) \cap F(r'_{ij})$. Hence, $q_i^T r_{ij} = (-q'_i)^T r_{ij} = q_i^T r'_{ij} = (-q'_i)^T r'_{ij} = 1$. This implies $(q_i - q'_i)^T (r_{ij} + r'_{ij}) = 4$, i.e. $t_i^T (r_{ij} + r'_{ij}) = 4$. Since $t_i^2 = 4 = (r_{ij} + r'_{ij})^2$, we obtain that $r_{ij} + r'_{ij} = t_i$. Similarly, the inclusion $e_j \in F(r_{ij}) \cap F(-r'_{ij})$ implies that $r_{ij} - r'_{ij} = t_j = q_j - q'_j$. So, we obtained the equalities in (4). \square

The four facets $F(\pm r_{ij}), F(\pm r'_{ij})$ of Proposition 1 form a quasi 4-belt which we denote by $B_{ij}(q)$. Each quasi 4-belt $B_{ij}(q)$ determines uniquely the four proper pairs $(\pm e_i, \pm e_j)$, and each proper pair of edges determines uniquely a quasi 4-belt.

Proposition 2. *Let t_i be a diagonal of $\beta_5(q)$. Then*

$$H_s(t_i) \cap P_V(E_6) = e_i,$$

where $e_i = (v(q_i), v(-q'_i)) \in E(q)$.

Proof. Recall that $v(\pm q)$ for $q \in \mathcal{M}$ are all vertices of $P_V(E_6)$. Hence the inequalities and equalities in (5) imply that $H_s(t_i)$ is a support hyperplane for $P_V(E_6)$, which contains only two vertices $v(q_i)$ and $v(-q'_i)$, i.e., it contains only the edge e_i . \square

Although the intersection $F(r_{ij}) \cap F(r'_{ij}) = e_i$ is 1-dimensional, the intersection $H_s(r_{ij}) \cap H_s(r'_{ij})$ of the supporting hyperplanes is, of course, 4-dimensional. Lemma 3 shows that this intersection contains the shifted set $T_{ij}(q)$.

The following is important for what follows. Let $X(G) \subseteq \mathcal{M}$ be a set of all vectors of \mathcal{M} that are parallel to m -edges of a k -face G of $P_V(E_6)$. Then the set $X(G)$ generates linearly the k -space supporting the face G .

Recall that a vector v is free for a parallelotope P if the Minkowski sum $P + [-v, v]$ is also a parallelotope.

Lemma 4. *There are the following 120 triples generating 6-belts of the Voronoi polytope $P_V(E_6)$, where facet vectors are given in the basis \mathcal{E} , $e(N_6) = \sum_{i \in N_6} e_i$, $e(S) = \sum_{i \in S} e_i$ and S is a subset of N_6 of cardinality 3.*

- (i) $e_i - e_j, e_j - e_k, e_k - e_i, i, j, k \in N_6$;
- (ii) $e(S), e(\overline{S}), e(N_6), \overline{S} = N_6 - S$;
- (iii) $e(S), e_i - e_j, e(S) - e_i + e_j, i \in S, j \notin S$.

Proof. It is easy to verify that there are 20 triples of type (i), 10 triples of type (ii) and 90 triples of type (iii), total 120 triples. Each 6-belt is uniquely determined by each of its six 4-faces. It is known (see, for example [CS91]) that $P_V(E_6)$ has 720 4-faces. Since each 4-face belongs exactly to one 6-belt, $P_V(E_6)$ has $\frac{720}{6} = 120$ 6-belts. \square

Note that the above 6-belts form 1-orbit under the action of the automorphism group of E_6 .

Call a vector *free for a triple* if it is orthogonal at least to one vector of the triple.

Proposition 3. *The Voronoi polytope $P_V(E_6)$ is free along a line l if and only if l is spanned by a minimal vector $q \in \mathcal{M}$ of the dual lattice E_6^* , described in (2).*

Proof. We seek a vector a which is free for $P_V(E_6)$ in the basis $\mathcal{A} = \{a_i : i \in N_6\}$ related to the lattice E_6^* which is dual to the basis $\mathcal{E} = \{e_i : i \in N_6\}$ related to the lattice E_6 . So, let $a = \sum_{i \in N_6} z_i a_i$ be a vector which is free for $P_V(E_6)$. We find conditions, when the vector a is orthogonal to at least one vector of each triple of types (i), (ii) and (iii) of Lemma 4. We shall see that a is, up to a multiple, one of the vectors $a_i, b_j, c_{kl}, i, j, k, l \in N_6$, of (1).

Claim 1. *The coordinates $z_i, i \in N_6$ take only two values. Suppose that there are three pairwise distinct coordinates $z_i \neq z_j \neq z_k \neq z_i$. Then the vector a is not free for a triple of type (i).*

So, a free vector has the form $a_T = za(T) + z'a(\overline{T})$, where $T \subseteq N_6, \overline{T} = N_6 - T$ and $a(T) = \sum_{i \in T} a_i$.

Claim 2. *If $z = 0$ or $z' = 0$, then $|\overline{T}| = 1$, or $|T| = 1$, respectively. In fact, if $z' = 0$ and $|T| \geq 2$, then the vector $a = za(T)$ is not free for each triple of type (ii) such that $S \cap T \neq \emptyset$ and $S \cap \overline{T} \neq \emptyset$.*

Note that Claim 2 implies that $T \neq \emptyset$ and $T \neq N_6$.

It is easy to verify that each of the six vectors a_i , $i \in N_6$, (which are minimal vectors of E_6^*) is free for all triples.

Now consider vectors $a_T = za(T) + z'a(\overline{T})$, with both non-zero coefficients z and z' .

Claim 3. $|T| \neq 3$. In fact, let $|T| = 3$. Consider a triple of type (ii) for $S = T$. The vector a_T is free for this triple only if $e(N_6)^T a_T = z + z' = 0$. Hence, a_T should take the form $a_T = z(a(T) - a(\overline{T}))$. But this vector is not free for a triple of type (iii) with $S = T$ and $i \in T$, $j \notin T$.

So, without loss of generality we can consider vectors a_T such that $|T| = 1, 2$.

Claim 4. $z = -2z'$. Let $|T| = 1$. For a_T to be free for a triple of type (ii) with $S \supset T$, the coefficients z, z' should satisfy one of the equalities $z + 2z' = 0$ or $z + 5z' = 0$. If $z = -5z'$, then a_T is not free for a triple of type (iii) such that $\{i\} = T$. Hence, $z = -2z'$. It is easy to verify that, for all $i \in N_6$, the vector $a_T = -z'(2a_i - a(N_6 - \{i\})) = -3z'(a_i - \frac{1}{3}a(N_6)) = -3z'b_i$ is free for all triples.

Now, let $|T| = 2$. For a_T to be free for a triple of type (ii) with $S \supset T$, the coefficients z and z' should satisfy one of the equalities $2z + z' = 0$ or $2z + 4z' = 0$. If $z' = -2z$, then the vector $a_T = z(a(T) - 2a(\overline{T}))$ is not free for a triple of type (iii) such that $S \cap T = \{i\}$ and $j \notin T$. One can verify that if $z = -2z'$ and $T = \{ij\}$, then, for $1 \leq i < j \leq 6$, the vector $a_T = 3z'(a_i + a_j - \frac{1}{3}a(N_6)) = 3z'c_{ij}$ is free for triples of all types. \square

6. WHEN THE SUM OF $P_V(E_6)$ WITH A ZONOTOPE IS A PARALLELOTOPE

For $U \subset \mathcal{M}$, set $Z(U) = \sum_{q \in U} [-q, q]$. Recall that $e(p)$ is an m -edge e that is parallel to a vector p .

It is proved in [Gr04] that if the sum $P + Z(U)$ of a parallelotope P and a zonotope $Z(U)$ is a parallelotope, then the vectors of U spans vectors of a unimodular system. Recall that a set of vectors U is called *unimodular* if, for any basic subset $B \subseteq U$, all vectors of U have integer coordinates in the basis B .

Note that

$$P_V(E_6) = \left\{ x \in \mathbb{R}^6 : r^T x \leq \frac{1}{2}r^2, r \in \mathcal{R} \right\}.$$

For each $r \in \mathcal{R}$, there are 6 pairs $p_1, p_2 \in \mathcal{M}$ such that $p_1 - p_2 = r$ and $p_1^T p_2 = \frac{1}{3}$.

Note that if t_i is a diagonal of $\beta_5(q)$, then $t_i = q_i - q'_i$ and $q = -(q_i + q'_i)$. Moreover, for any $p, q \in \mathcal{M}$ such that $p^T q = -\frac{2}{3}$, the vector $p - q$ is a diagonal t of the cross-polytope $\beta_5(-(p + q))$. Let

$$\mathcal{T} = \left\{ t = p - q : p^T q = -\frac{2}{3}, p, q \in \mathcal{M} \right\}$$

be the set of all diagonals of all cross-polytopes β_5 . Note that the same diagonal is represented by two vectors t and $-t$.

Let $t = p - q \in \mathcal{T}$. It is easy to verify that $t^2 = 4$, $t^T p' \leq \frac{1}{2}t^2$ for all $p' \in \pm \mathcal{M}$ with equality if $p' = p$ or $p' = -q$. Since vertices of $P_V(E_6)$ are $v(p'), v(-p')$ for $p' \in \mathcal{M}$, we have

$$P_V(E_6) \subseteq \left\{ x \in \mathbb{R}^6 : t^T x \leq \frac{1}{2}t^2, t \in \mathcal{T} \right\}.$$

Hence, we have

$$P_V(E_6) = P(a_0),$$

where $a_0 = a_0(p, q) = \frac{1}{2}(p - q)^2$,

$$(6) \quad P(a) = \{x \in \mathbb{R}^6 : (p - q)^T x \leq a(p, q), \ p, q \in \mathcal{M}\},$$

and $a(p, q) = a(q, p)$ is a symmetric function on pairs $p, q \in \mathcal{M}$. If $a(p, q) = a_0(p, q)$, then the inequalities in (6) for p, q such that $p^T q = -\frac{2}{3}$ are redundant.

Call a subset $X \subset \mathcal{M}$ of cardinality 5 *forbidden* if $p^T p' = \frac{1}{3}$ for all $p, p' \in X$, $p \neq p'$. By Lemma 1, any forbidden set has the form $T_J(q)$ (see (3)) for some $q \in \mathcal{M}$ and $J \subseteq I_5$. Under the action of the automorphism group of \mathcal{M} there are two orbits of forbidden subsets; they correspond to the 2 orbits of 4-faces of P_{Sch} .

We show below that if U does not contain a forbidden set, then $P_V(\mathbf{E}_6) + Z(U) = P(a_U)$, where some inequalities with $p^T q = -\frac{2}{3}$ are necessary.

At first, we prove that $P_V(\mathbf{E}_6) + Z(U)$ is not a parallelotope if U contains a forbidden set. In order to prove this, we need the following.

Recall that $T_{ij}(q) = T_i(q) \cap T_j(q)$, and $P_V(\mathbf{E}_6)$ has ten edges $\pm e_i \in E(q)$, $i \in I_5$, which are parallel to $q \in \mathcal{M}$. For a set of vectors U , let $\dim(U)$ be dimension of the space spanned by U .

Lemma 5. *Let $e_i(q) = (v(q_i), v(-q'_i)) \in E(q)$, $U \subseteq \mathcal{M}$ and $U_i = U \cap T_i(q)$. Then the edge $e_i(q)$ is transformed in the sum $P_V(\mathbf{E}_6) + Z(U)$ into a zonotopal face $G_i = e_i(q) + Z(U_i)$ of dimension $\dim(U_i \cup \{q\})$. This face G_i lies in the intersections $H_s(r_{ij}) \cap H_s(r'_{ij})$ for all $j \in I_5$ such that $U_i \subseteq T_{ij}(q)$.*

Proof. For all $p \in \mathcal{M} - \{q_i, q'_i\}$, consider the lines $l(p)$ touching the edge $e_i(q)$. It is easy to verify that each line $l(p)$ lies either on the surface of $P_V(\mathbf{E}_6)$ or in the hyperplane $H_s(t_i)$. Hence, by Lemma 2 and Proposition 2, the sums $e_i(q) + [-p, p]$ generate new faces only if $p \in T_i(q) - \{q\}$. Since e_i is parallel to q , dimension of G_i is equal to $\dim(U_i \cup \{q\})$. Lemma 3 implies that the face G_i lies in the intersections $H_s(r_{ij}) \cap H_s(r'_{ij})$ for all $j \in I_5$ such that $q_j, q'_j \notin U_i$. \square

Proposition 4. *Let $U \subseteq \mathcal{M}$. The polytope $P(U) = P_V(\mathbf{E}_6) + Z(U)$ is not a parallelotope if U contains a forbidden set.*

Proof. Recall, that every forbidden subset $U \subset \mathcal{M}$ has the form $T_J(q) = \{q_i : i \in J\} \cup \{q'_i : i \in J'\}$, where $q \in \mathcal{M}$ and $J \cup J' = I_5$. The set $T_J(q)$ contains five vectors by one from each pair $\{q_i, q'_i\}$.

It is sufficient to prove that $P(U)$ is not a parallelotope if $U = T_J(q)$. Recall that $T_i(q) = \{q\} \cup \{q_j, q'_j : j \in I_5, j \neq i\}$. For $e_i \in E(q)$ and $U = T_J(q)$, the set $U_i = U \cap T_i(q)$ consists of four linearly independent vectors. Hence, $\dim(U_i) = 4$. Since $q \notin U_i$, by Lemma 5, the edge $e_i \in E(q)$ is transformed into a face $G_i = e_i + Z(U_i)$ of dimension $\dim(U_i \cup \{q\}) = 5$. Hence, G_i is a facet. This assertion is true for all $i \in I_5$, i.e., for all ten edges $\pm e_i$, $i \in I_5$, of the set $E(q)$. Recall that each proper pair of edges e_i, e_j from $E(q)$ belongs to the quasi 4-belt $B_{ij}(q)$, which contains also the edges $-e_i, -e_j$. Since each of these four edges is transformed into a facet, $B_{ij}(q)$ is transformed into a belt with 8 facets of the polytope $P(U) = P_V(\mathbf{E}_6) + Z(U)$. This is not possible for a parallelotope. This implies that U does not contain a forbidden set if $P(U)$ is a parallelotope. \square

Consider the zonotope $Z(U)$, where $U \subseteq \mathcal{M}$, in more details. Among many remarkable properties of the set \mathcal{M} , there is one that can be verified by inspection. It is described in Lemma 6 below.

Lemma 6. *Let $X \subset \mathcal{M}$ be a non-forbidden subset of 5 linearly independent vectors. Then there are two vectors $p, q \in \mathcal{M}$ such that the hyperplane linearly generated by vectors of X is orthogonal to $p - q$, i.e. this hyperplane is $H(p - q)$. \square*

For simplicity sake, call a hyperplane H *supporting* for a face G (or H *supports* G) if a parallel shift of H contains G .

Proposition 5. *Let U does not contain a forbidden set. Then $P_V(E_6) + Z(U) = P(a_U)$, where $a_U(p, q) \geq \frac{1}{2}(p - q)^2$ and $P(a)$ is defined in (6).*

Proof. Lemma 6 implies that facets of the zonotope $Z(U)$ are supported by hyperplanes $H(p - q)$ for some $p, q \in \mathcal{M}$. Since $Z(U)$ is centrally symmetric we have $Z(U) = P(a'_U)$ for some positive symmetric function $a'_U(p, q)$ depending on the set $U \subseteq \mathcal{M}$. Since $P_V(E_6) = P(a_0)$ and $Z(U) = P(a'_U)$, we have $P_V(E_6) + Z(U) = P(a_U)$, where $a_U = a_0 + a'_U$. Since $a'_U \geq 0$, we have $a_U(p, q) \geq a_0(p, q) = \frac{1}{2}(p - q)^2$. \square

Now we prove the main result.

Theorem 2. *For $U \subseteq \mathcal{M}$, the polytope $P_V(E_6) + Z(U)$ is a parallelotope if and only if U does not contain a forbidden set.*

Proof. We have to know when the sum $P_V(E_6) + Z(U)$ is a parallelotope. By Proposition 4, U must not contain a forbidden set for this sum to be a parallelotope. By Proposition 5, in this case, $P_V(E_6) + Z(U) = P(a_U)$.

Obviously, $P(a_U)$ and all its facets are centrally symmetric. Hence, we have to know when $P(a_U)$ has only 4- and 6-belts.

Each belt B of the polytope $P(a_U)$ is determined by a set of mutually parallel 4-faces. The facets of the belt B are mutually contiguous by these 4-faces. Hence, we have to study 4-faces of the polytope $P(a_U)$. Let G be a 4-face of $P(a_U)$. The m -edges of G are parallel to vectors of a subset $Y \subset \mathcal{M}$. The vectors of Y linearly generate a 4-space $H(G)$ that supports the face G . In particular, there is a subset $X \subseteq Y$ of four linearly independent vectors that generates the space $H(G)$.

It is convenient to describe a set $X \subset \mathcal{M}$ of four linearly independent vectors by a graph Γ on four vertices $p \in X$. Two vertices p, p' of Γ are adjacent by an edge if and only if $p^T p' = -\frac{2}{3}$.

The group of automorphisms of \mathcal{M} is such that sets X with the same graph Γ are equivalent under actions of the group. Note that a triangle represents 3 dependent vectors. There are the following 6 graphs on sets of four linearly independent vectors of \mathcal{M} :

- an empty graph Γ_0 without edges;
- a graph Γ_1 with one edge;
- a graph Γ_2 with two adjacent edges;
- a graph Γ_3 with three edges with a common vertex;
- a graph Γ'_3 with three edges forming a path;
- a graph Γ_4 with four edges forming a 4-circuit.

Denote the 4-space generated by vectors of a 4-set X with a graph Γ by H_Γ . Obviously, H_Γ is an intersection of hyperplanes $H(r)$ and $H(t)$ for some $r \in \mathcal{R}$ and $t \in \mathcal{T}$. One can verify that there are the following three types of intersections of dimension four:

- $H_{\Gamma_0} = H(r) \cap H(t_1) \cap H(t_2)$, where $r + t_1 + t_2 = 0$;

- $H_{\Gamma_1} = H_{\Gamma_3} = H(r) \cap H(r') \cap H(t) \cap H(t')$, where $r^T r' = t^T t' = 0$ and $r + r' = t, r - r' = t'$;
- $H_{\Gamma_2} = H_{\Gamma'_3} = H_{\Gamma_4} = H(r_1) \cap H(r_2) \cap H(r_3)$, where $r_1 + r_2 + r_3 = 0$.

Obviously, 4-faces, whose supporting spaces lie in the intersections of the first and third types, generate 6-belts.

A 4-face, whose supporting space lies in the second intersection, generates the belt $B_{ij}(q)$, where the vector $q \in \mathcal{M}$ is such that $t = t_i(q)$ and $t' = t_j(q)$ (see (4) and Lemma 2). The belt $B_{ij}(q)$ is a 8-belt if both the hyperplanes $H(t)$ and $H(t')$ support facets.

Let $F(t)$ be a face supported by $H(t)$. The proof of Proposition 4 shows that for each pair ij and each $q \in \mathcal{M}$, at least one of faces $F(t_i)$ and $F(t_j)$ of the belt $B_{ij}(q)$ is not a facet if U does not contain a forbidden set. Hence, each belt $B_{ij}(q)$ is at most a 6-belt. \square

Note that the proof of Theorem 2 does not use unimodularity of the set U . This means that any set $U \subseteq \mathcal{M}$ that does not contain a forbidden set is unimodular.

7. MAXIMAL FEASIBLE SUBSETS IN \mathcal{M}

It is convenient to denote a unimodular system U by a regular matroid M_U , which is represented by U . There are many definitions of a matroid, see, for example, [Aig79]. In particular, a matroid on a ground set X is a family \mathcal{C} of *circuits* $C \subseteq X$ satisfying the following axioms:

- If $C_1, C_2 \in \mathcal{C}$, then $C_1 \not\subseteq C_2, C_2 \not\subseteq C_1$
- and if $x \in C_1 \cap C_2$, then there is $C_3 \in \mathcal{C}$ such that $C_3 \subseteq C_1 \cup C_2 - \{x\}$.

Note that linear dependencies between vectors of any family of vectors satisfy these axioms. A matroid M on a set X is *represented* by a set of vectors U if there is a one-to-one map $f : X \rightarrow U$ such that, for all $C \in \mathcal{C}$, the set $f(C)$ is a minimal by inclusion linearly dependent subset of U .

Each unimodular set of vectors represents a *regular* matroid. Special cases of a regular matroids are *graphic* $M(G)$ and *cographic* $M^*(G)$ matroids whose ground sets are the set of edges of a graph G . The families of circuits of these matroids are *cycles* and *cuts* of the graph G , respectively.

Seymour proved in [Se80] that a regular matroid is a 1-, 2- and 3-sum of graphic, cographic matroids and a special matroids R_{10} , which is neither graphic nor cographic (see also [Tr92]). Using this work of Seymour, the authors of [DG99] described maximal by inclusion unimodular systems of a given dimension. Their description is similar to that of Seymour, but they denote a k -sum of Seymour by $(k-1)$ -sum, and slightly changed the definition of the k -sum. Namely, k -sum of Seymour gives the symmetric difference of the summing sets, but the corresponding $(k-1)$ -sum of [DG99] gives the union of the summing sets. Below, we use the second k -sum from [DG99].

The graphic matroid $M(K_{n+1})$ of the complete graph K_{n+1} on $n+1$ vertices is represented by the root system A_n , which is one of maximal by inclusion unimodular n -dimensional systems. The matroid R_{10} is represented by a ten-element unimodular system of dimension 5, which is denoted in [DG99] by E_5 . Each maximal by inclusion cographic unimodular system of dimension n represents the cographic matroid $M^*(G)$ of a 3-connected cubic non-planar graph G on $2(n-1)$ vertices and $3(n-1)$ edges.

nr	$ U $	$\dim U$	$ Stab U $	status	ref
1	9	5	384	graphic	$M(G_1)$
2	12	5	96	cographic	$M^*(G_2)$
3	12	6	12	special	R_{12}
4	12	6	12	special	$R_{10} \oplus_1 C_3$
5	13	6	4	cographic	$M^*(G_5)$
6	13	6	4	cographic	$M^*(G_6)$
7	13	6	2	cographic	$M^*(G_7)$
8	14	6	8	graphic	$M(G_8)$
9	14	6	24	graphic	$M(G_9)$
10	15	6	24	cographic	$M^*(G_{10})$

TABLE 3. All ten maximal admissible unimodular system of \mathcal{M}

b_2	1	0	0	0	0	0	b_2	1	0	0	0	0	0
c_{26}	0	1	0	0	0	0	c_{26}	0	1	0	0	0	0
a_2	0	0	1	0	0	0	a_2	0	0	1	0	0	0
a_3	0	0	0	1	0	0	a_3	0	0	0	1	0	0
b_1	0	0	0	0	1	0	b_1	0	0	0	0	1	0
b_5	0	0	0	0	0	1	b_5	0	0	0	0	0	1
b_3	1	0	-1	1	0	0	b_3	1	0	-1	1	0	0
c_{36}	0	1	1	-1	0	0	c_{36}	0	1	1	-1	0	0
a_4	0	1	0	-1	-1	-1	a_4	0	1	0	-1	-1	-1
a_1	-1	0	1	0	1	0	a_1	-1	0	1	0	1	0
c_{14}	0	-1	0	1	0	1	c_{16}	1	1	0	0	-1	0
c_{15}	1	0	-1	0	-1	-1	c_{45}	0	-1	0	1	1	0

The unimodular system R_{12} The unimodular system $R_{10} \oplus_1 C_3$

TABLE 4. Unimodular systems which are neither graphic nor cographic

All ten maximal by inclusion unimodular feasible subsets $U \subseteq \mathcal{M}$ were found by computer. The method is to take an unimodular subset, to consider all possible ways to extend it in an acceptable way and to reduce by isomorphism. The results are presented on Table 3. We give vectors of these sets in the denotations (2). Let $\text{Aut}(\mathcal{M})$ be the group of automorphisms of the set \mathcal{M} . We denote by $\text{Stab}(U) \subseteq \text{Aut}(\mathcal{M})$ the stabilizer subgroup of the set U . Among these 10 sets there are two absolutely maximal 6-dimensional unimodular sets U_3 and U_{10} (see [DG99]). The matroids which are neither graphic nor cographic are given in Table 4 and the rest in Figure 2.

8. RESULTS FOR OTHER LATTICES

By using the algorithms explained in Section 2 one proves that the polytopes $P_V(L)$ are nonfree for $L = E_7^*$, κ_8^* , κ_9^* , Λ_{10}^* or K_{12} . For the lattice BW_{16} this direct approach does not work, because there are too many 6-belts. However, by selecting a subset of the 6-belts, one can prove that this lattice is nonfree as well.

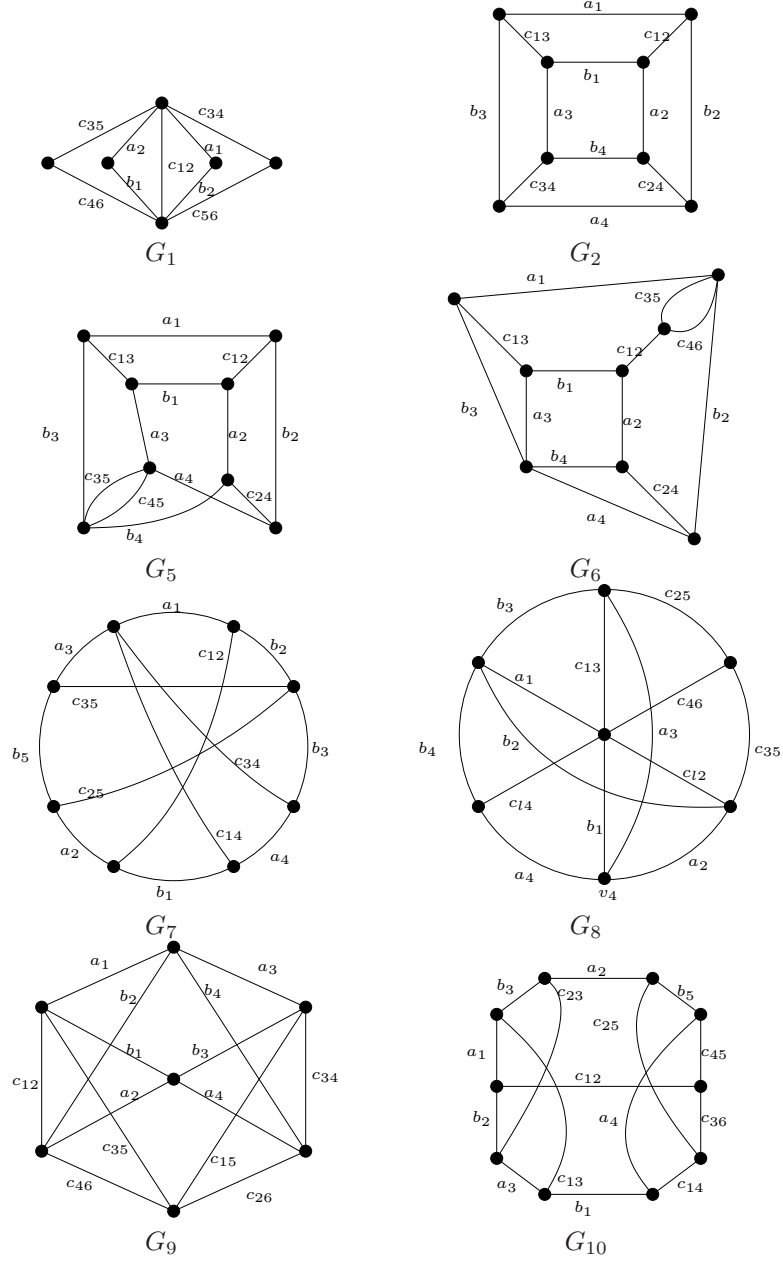


FIGURE 2. Occurring graphs

For a general lattice L , we cannot expect a simple criterion for determining the forbidden and feasible subsets of $P_V(L)$. This is because the lattice E_6 is very symmetric and other cases are necessarily more complicated.

In Table 5 we give detailed information on 12 lattices of the computed data. For the lattice ER_7 , we found out that there exist some minimal forbidden subsets U

L	$ \mathcal{F}(L) $	$ O(\mathcal{F}_{min}(L)) $	$ O(\mathcal{F}_{max}(L)) $	$\dim_{max}(L)$	$S_{max}(L)$
E_6	27	2	10	6	15
E_7	28	2	4	7	14
ER_7	28	32	99	7	12
ER_7^*	6	0	1	6	6
κ_7	11	2	2	6	9
κ_7^*	10	2	1	1	1
κ_8	6	1	2	3	3
κ_9	3	1	1	2	2
Λ_9	1	0	1	1	1
Λ_9^*	16	2	1	1	1
Λ_{10}	3	0	1	2	3
O_{10}	40	2	2	5	10

TABLE 5. The size of $\mathcal{F}(L)$, the number of orbits in $\mathcal{F}_{min}(L)$, respectively $\mathcal{F}_{max}(L)$, the maximal rank of a feasible subset and the maximal size of a feasible subset for 12 lattices. The lattices are given in [NeSo12] and the lattice ER_7 is given in [ErRy02].

such that $P_V(ER_7)$ has some faces G such that $\dim G_U = 7$. Hence, for this case, one cannot limit oneself to the quasi 4-belt in the analysis.

9. ACKNOWLEDGMENTS

This article is dedicated to the memory of the late Evgenii Baranovskii (died in March 2012).

REFERENCES

- [Aig79] M. Aigner, Combinatorial Theory, Springer-Verlag, 1979.
- [Ba91] E.P. Baranovskii, Partition of Euclidean spaces into L -polytopes of some perfect lattices, Proc. Steklov Inst. Math. 196 (1991) 29–51.
- [Barn57] E.S. Barnes, The complete enumeration of extreme senary forms, Phil. Trans. Roy. Soc., London A249 (1957) 461–506.
- [Burn11] W. Burnside, Theory of groups of finite order, Cambridge Univ. Press, 1911.
- [CS91] J.H. Conway, N.J.A. Sloane, The cell structures of certain lattices, in: Miscellanea Mathematica, Springer-Verlag, (1991) 71–107.
- [Cox83] H.S.M. Coxeter, The twenty-seven lines on the cubic surface, in: Convexity and its Applications, Birkhäuser Verlag, (1983) 111–119.
- [DG99] V.I. Danilov, V.P. Grishukhin, Maximal unimodular systems of vectors, Eur. J. Combin. 20 (1999) 507–526.
- [Du10] M. Dutour, polyhedral, <http://www.liga.ens.fr/~dutour/polyhedral/index.html>
- [DSV09] M. Dutour Sikirić, A. Schürmann, F. Vallentin, Complexity and algorithms for computing Voronoi cells of lattices, Math. Comput. 78 (2009) 1713–1731.
- [DV05] M. Dutour, F. Vallentin, Some six-dimensional rigid lattices, Proceedings of “Third Voronoi Conference of the Number Theory and Spatial Tessellations”, 102–108.
- [DSV08] M. Dutour Sikirić, A. Schürmann, F. Vallentin, A generalization of Voronoi’s reduction theory and applications, Duke Math. J. 142 (2008) 127–164.
- [En98] P. Engel, Investigations of parallelohedra in \mathbb{R}^d , in: Voronoi’s Impact on Modern Science, vol 2, eds. P. Engel and H. Syta, Institute of Math., Kyiv, (1998) 22–60.
- [En00] P. Engel, The contraction types of parallelohedra in E^5 , Acta Cryst. Ser. A 56 (2000) 491–496.
- [ErRy94] R.M. Erdahl, S.S. Ryshkov, On lattice dicings, Eur. J. Combin. 15 (1994) 459–481.

- [Er99] R.M. Erdahl, Zonotopes, Dicings, and Voronoi's conjecture on Parallelehedra, Eur. J. Combin. 20 (1999) 527–549.
- [ErRy02] R.M. Erdahl, K. Rybnikov, Voronoi-Dickson Hypothesis on Perfect Forms and L-types, Peter Gruber Festschrift: Rendiconti del Circolo Matematico di Palermo, Serie II, Tomo LII, part I (2002) 279–296.
- [Gr04] V.P. Grishukhin, Parallelotopes of non-zero width, Math. Sbornik 195 (2004) 59–78, translated in: Sb. Math. 195 (2004) 669–686.
- [Gr06a] V.P. Grishukhin, Free and Nonfree Voronoi Polyhedra (in Russian), Matematicheskie Zametki 80 (2006) 367–378, translated in Math. Notes 80 (2006) 355–365.
- [Gr06b] V.P. Grishukhin, The Minkowski sum of a parallelootope and a segment (in Russian), Math. Sbornik 197 (2006) 15–32, translated in: Sb. Math. 197 (2006) 1417–1433.
- [Hu90] J.E. Humphreys, Reflection groups and Coxeter groups, Cambridge University Press, 1990.
- [Mu80] P. McMullen, Convex bodies which tile space by translation, Mathematika 27 (1980) 113–121.
- [NeSo12] G. Nebe, N.J.A. Sloane, A catalog of lattices, <http://www2.research.att.com/~njas/lattices/>
- [Re88] M. Reid, Undergraduate algebraic geometry, Lond. Math. Soc. student texts, Cambridge Univ. Press, 1988.
- [Sch09] A. Schürmann, Computational geometry of positive definite quadratic forms, University Lecture Notes, AMS, 2009.
- [Se80] P.D. Seymour, Decomposition of regular matroids, J. Comb. Theory ser. B 28 (1980) 305–359.
- [Sha88] I.R. Shafarevich, Foundations of algebraic geometry Vol. 1 (in Russian), Nauka, Moscow, 1988, translated in: Springer Verlag, 1994.
- [Tr92] K. Truemper, Matroid decomposition, Academic Press, 1992.
- [Ve54] B.A. Venkov, On a class of Euclidean polytopes (in Russian), Vestnik Leningradskogo Univ., Ser. Math. Phys. Chem. 9 (1954) 11–31.
- [Vo08] G.F. Voronoi, Nouvelles applications des paramètres continus à la théorie des formes quadratiques, Deuxième Mémoire, Recherches sur les paralléloèdres primitifs, J. Reine Angew. Math. 134 (1908) 198–287 and 136 (1909) 67–181.
- [Zo96] C. Zong, Strange phenomena in convex and discrete geometry, Universitext, Springer-Verlag, New York, 1996.

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